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Translated by M.D.F.

PMM U.S.S.R., Vol.53, No.4, pp.495-500, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00


# A VARIATIONAL PRINCIPLE FOR NON-LINEAR CONCENTRATED WAVES*

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Asymptotic solutions describing domain walls and wave beams in a non-linear continuous medium are considered. The shape of the walls or beams can be derived from a simple variational principle - a generalization of Fermat's principle in linear geometrical optics to the non-linear situation.

*1. Statement of the problem.* We consider asymptotic solutions for certain classes of non-linear equations. The following equations will be studied:

$$\Delta u + \omega^2 V_u'(u, \mathbf{x}) = 0, \quad u(\mathbf{x}): R^m \to R, \quad \mathbf{x} \in R^m \tag{1.1}$$

and also

$$\Delta u + \omega^2 \Phi'_{|u|^2}(|u|^2, \mathbf{x}) u = 0, \quad u(\mathbf{x}) \in C, \quad \mathbf{x} \in R^k \tag{1.2}$$

where $\omega \gg 1$, $m \geqslant 1$, $k = 2, 3$.

Eq.(1.1) has applications (when $m = 2$) in two-dimensional problems of elasticity theory for liquid crystals (these applications will be considered in Sect.4). Eq.(1.2) is used to describe the propagation of radiation in a non-linear medium /1, 2/. In that case $u$ is the complex amplitude of the field and $\Phi'$ is the non-linear refractive index.

Special asymptotic solutions (as $\omega \to \infty$) of Eqs.(1.1), (1.2) were considered in /3/. In this paper, for brevity, we shall use the term "concentrated solution" (for a rigorous definition see /3/).

The following is an example of a concentrated solution. Put $m = 1$, $\alpha > 0$, $V_u' = \alpha^2(x) \sin u$ in (1.1). Then the equation has an asymptotic solution

$$u = 4 \operatorname{arctg}(\exp(\omega\alpha(x_0)(x - x_0)) + O(\omega^{-1}) = u_0(x) + O(\omega^{-1}) \tag{1.3}$$

The function $u_0(x)$ varies essentially in a narrow region, of size $O(\omega^{-1})$, near the point $x_0$, but when $|x - x_0| \gg \omega^{-1}$ the function $u_0(x)$ differs by an exponentially small amount from 0 or $2\pi$. The solution is concentrated near $x_0$. When $m > 1$ such solutons of Eq.(1.1) are concentrated near hypersurfaces $S$ in $R^m$; similar solutions of Eq.(1.2) concentrate near curves $l$. Solutions of Eq.(1.1) are interpreted as domain walls, and those of

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*Prikl.Matem.Mekhan., 53, 4, 636-641, 1989

Eq.(1.2) as wave beams.

The main aim of this paper is to formulate a simple variational principle by means of which it is possible, on the sole basis of the form of the equations, to determine the above-mentioned hypersurfaces and curves, avoiding the need to determine the principal term $u_0$ of the asymptotic expansion. The point is that an explicit expression for this principal term does not always exist. Nevertheless, it is often possible to find the region in which the solution is concentrated without using formulae for $u_0$. Our variational principles are non-linear generalizations of Fermat's principle in geometrical optics. In the case of (1.1), they are valid for any functions $V(u, x)$ satisfying Condition 1 (see below, Sect. 2). For Problem (1.2), if $x \in R^3$, we have been able to obtain a simple Fermat-type principle only for potentials of the form

$$\Phi = 2\alpha(\mathbf{x}) \mid u \mid^{p+2} (p+2)^{-1} + \beta(\mathbf{x}) \mid u \mid^2, \ p > 0, \ \alpha, \ \beta > 0 \tag{1.4}$$

*2. Asymptotic formalism for Eq.(1.1).* For an arbitrary function $V$ a concentrated solution of Eq.(1.1) need not always exist. The following simple conditions, whose meaning will be clarified later, are sufficient for that to be the case:

*Condition 1.* $V \in C^\infty$, $|V| < C_1$, and for every $\mathbf{x}$ the function $V$ has two local maxima $u_+(\mathbf{x})$, $u_-(\mathbf{x})$, such that

$$V_{\max} = V(u_+(\mathbf{x}), \mathbf{x}) = V(u_-(\mathbf{x}), \mathbf{x}), \quad \min_{\mathbf{x}} | V_{uu}(u_+(\mathbf{x}), \mathbf{x}) | > \delta > 0 \tag{2.1}$$

*Condition 2.* For all local maxima of $V$ other than $u_+$, $u_-$, the value of $V$ at these points does not exceed $V_{\max} - \delta_1$ $(\delta_1 > 0)$.

We may assume without loss of generality that $V_{\max} = 0$. To simplify the constructions, we shall also assume that the functions $u_\pm$ are independent of $x$.

The method is analogous to that of /4/, except that the sufficient conditions considered there in the case $m = 2$ are different.

We shall look for a solution of Eq.(1.1) concentrated in an $O(\omega^{-1})$-neighbourhood of a smooth hypersurface $S \subset R^m$. This hypersurface must be a closed manifold. In a neighbourhood $\Omega(M_1)$ of a point $M_1 \in S$, we introduce coordinate $n$, $s$, where the coordinates $s = (s_1, s_2, s_3, \ldots, s_{m-1})$ parametrize $\Omega(M_1) \cap S$, and $|n|$ is the distance from $M$ to its projection on $S$. The sign of $n$ is chosen depending on which side of $S$ the point $M$ is situated. For example, $\Omega$ may be chosen as a sphere of diameter $O(\omega^{-1/2})$. Then the coordinates just defined do not degenerate as $\omega \to \infty$.

In the neighbourhood $\Omega$ we construct a solution

$$u = u_0(\nu, s) + \omega^{-1} u_1(\nu, s) + \ldots, \ \nu = \omega n \tag{2.2}$$

The Laplacian $\Delta u$ of (1.1) may be expressed in terms of the coordinates $n$, s, up to terms in $\omega^2$, $\omega$, as follows:

$$\Delta u = \omega^2 u''_{\nu\nu} + \omega\varkappa(s) u_\nu' + O(1) \tag{2.3}$$

In order to determine $\varkappa(s)$, replace $u$ in (2.3) by a linear function of $\nu$. The details may be found in /5/. It turns out that $\varkappa$ equals the mean curvature of $S$ (for the definition of mean curvature, see /6, p.30/). Now, expanding $V_u'$ (see (1.1) in powers of $u$ and $n$, we obtain an equation for the principal term:

$$\partial^2 u_0 / \partial\nu^2 + V_u'(u_0, 0, s) = 0, \quad V(u, 0, s) = V(u, \mathbf{x})|_{n=0} \tag{2.4}$$

The corrections $u_j$ $(j = 1, 2, \ldots)$ are obtained from recurrence relations

$$Q u_j = F_j(u_0, u_1, \ldots u_{j-1}, s) \quad (Q = \partial^2/\partial\nu^2 + V''_{uu}(u_0, 0, s)) \tag{2.5}$$

It can be shown that

$$F_1 = -\varkappa u'_{0\nu} - \nu V''_{un}(u_0, n, s) \tag{2.6}$$

Eqs.(2.4) and (2.5) must be considered together with boundary conditions that yield a concentrated solution:

$$\nu \to \pm\infty, \quad u_0(\nu, s) \to u_\pm, \quad u_j(\nu, s) \to 0 \tag{2.7}$$

An explicit solution $u_0(\nu, s)$ of Problem (2.4), (2.7) is given by the formula

$$\nu + h(s) = \int_{u_-}^{u_0} (-2V(z, 0, s))^{-1/2} dz, \quad u_- < u_0 < u_+ \tag{2.8}$$

Conditions 1 and 2 imply that the expression under the radical is positive, and we obtain asymptotic relations

$$\nu \to \pm\infty, \quad |u_0 - u_\pm| = O(\exp(-\delta_2 |\nu|)), \quad \delta_2 > 0.$$

The constant $h(s)$ can be determined by an additional condition, e.g., $0 = \mathrm{argmax}_\nu |u_{0\nu}'(\nu, s)|$.

We now consider the equations for the corrections (2.5). A direct check shows that the kernel of the operator $Q$ with zero boundary conditions at infinity is not empty: $Qu_{0\nu}' = 0$. Hence it follows that a bounded correction $u_j$ exists only if

$$\int_{-\infty}^{\infty} F_j(\nu, s) u_{0\nu}' \, d\nu = 0 \tag{2.9}$$

In particular, for $j = 1$ we obtain from (2.6) and (2.9)

$$\varkappa = -\frac{a_1}{a_0}, \quad a(n, s) = \int_{-\infty}^{\infty} V(u_0(\nu), n, s) \, d\nu \tag{2.10}$$

$$n = 0, \quad a_1 = a_n', \quad a_0 = a$$

These relations admit of a simple geometrical interpretation (see below, Sect.3) and impose restrictions on the surface $S$. If condition (2.9) holds with $j = 1$, then

$$u_1(\nu, s) = C_1(s) u_{0\nu}'(\nu, s) + u_{0\nu} \int_0^{\nu} u_{0\eta}'^{-2} \, d\eta \int_{-\infty}^{\eta} F_1(\xi) \, d\xi \tag{2.11}$$

The constant $C_1(s)$ is determined only at the second step, when $u_2$ is being determined, on the basis of the truth of (2.9) for $j = 2$. Similarly, Conditions (2.9), $j \geqslant 2$, may always be satisfied by a suitable choice of $C_{j-1}(s)$. At step $j$, as follows from (2.9):

$$C_j \int_{-\infty}^{\infty} u_{0\nu}'^2 \, d\nu = B_j(s)$$

where $B_j$ is a function depending on the previously found corrections $u_0, u_1, \ldots, u_{j-1}$.

Proceeding in this way one can determine an unlimited number of corrections $u_j$, with the following estimate holding at each step:

$$\nu \to \pm\infty, \quad |u_j| = O(\exp(-\delta_3 |\nu|))$$

We have thus constructed an asymptotic solution in the neighbourhood $\Omega(M_1)$. We note that neither $u_0(\mathbf{x})$ nor $u_1(\mathbf{x})$ depend on the choice of the coordinates s in $\Omega$. It is thus possible to extend the solution to a $O(\omega^{-1/2})$ -neighbourhood $\Omega(S)$ of the entire hypersurface $S$, "matching" solutions for different neighbourhoods $\Omega(M_1)$. Outside $\Omega(S)$ we can determine $u(\mathbf{x}) \equiv u_+$ or $u(\mathbf{x}) \equiv u_-$, depending on which side of $S$ the point $\mathbf{x}$ is situated. Then the two solutions (outside $\Omega(S)$ and in it) can be combined by a partition of unity. As a result we obtain an asymptotic solution $\bmod (\omega^{-n})$, $n \in N$.

*Remark about Conditions 1 and 2.* If one relaxes the condition $V(u_+, \mathbf{x}) \equiv V(u_-, \mathbf{x})$, Eq. (1.1) need not have a concentrated solution. However, if this equality holds only on a certain smooth curve $\Gamma$ (let $m = 2$), then a solution may nevertheless be constructed, subject to certain non-degeneracy conditions on $\Gamma$ /4/. The solution is concentrated in the neighbourhood of $\Gamma$, and the orthogonality Condition (2.9) is satisfied by suitable choice of $h(s)$. A rigorous discussion can be found in /4/.

If, however, the function $V(u_+, \mathbf{x})$ is everywhere different from $V(u_-, \mathbf{x})$ (where $u_\pm$ are the only local maxima of $V$), one can construct asmyptotic solutions of the parabolic equation

$$u_t = \Delta u + \omega^2 V_u'(u, \mathbf{x}) \tag{2.12}$$

which is a multidimensional generalization of Fisher's equation /7/. However, these solutions are explicitly time-dependent. Eq.(2.12) in the case $m = 1$ was studied in /8/.

## 3. *Variational principle for the surface S.*

We will now explain the geometrical meaning of Conditions (2.10). It follows from (2.8) that $d\nu = (-2V(u, 0, s))^{-1/2} du$. Transforming the integral for $a$ to the variable $u$, we obtain

$$a(\mathbf{x}) = \text{const} \int_{u_-}^{u_+} \sqrt{-V(u, \mathbf{x})}\, du \tag{3.1}$$

Thus, $a(\mathbf{x})$ is defined directly in terms of Eq.(1.1), and (3.1) can be used to extend it from $\Omega$ to the whole space $R^m$. It turns out that $S$ satisfies the following variational principle:

$$\delta L = \delta \int_S a(\mathbf{x})\, d\sigma = 0 \tag{3.2}$$

where $d\sigma$ is the differential form of the $(m-1)$-dimensional volume (measure) on $S$.

The geometric meaning of (3.2) is that $S$ is a minimum surface in the conformally Euclidean metric $a(\mathbf{x}) \langle\ \rangle$. The physical interpretation is also simple: if the solution is interpreted as a domain wall, Condition (3.2) means that the contribution of the energy of the domain wall to the energy of the system is extremal.

It can be proved that (3.2) and (2.10) are equivalent.

Let $S$ be an extremal of (3.2). Subject $S$ to an infinitesimal deformation by displacing each point along the normal to $S$ for a distance $\delta n(\mathbf{s})$. The corresponding increment to the measure of $S$ is /6/

$$\delta W = -\int_S \delta n(\mathbf{s})\, \varkappa(\mathbf{s})\, d\sigma$$

The increment $\delta L$ of the functional $L$ splits into two parts: a contribution $\delta L_1$ due to the increment of the volume of $S$ and a contribution $\delta L_2$ due to the change in $a(\mathbf{x})$. We have /6/

$$\delta L_1 = -\int_S \varkappa(\mathbf{s})\, a_0(\mathbf{s})\, \delta n(\mathbf{s})\, d\sigma, \quad \delta L = \int_S (a_1(\mathbf{s}) - \varkappa a_0(\mathbf{s}))\, \delta n(\mathbf{s})\, d\sigma = 0$$

This equation is equivalent to (2.10), since $\delta n(\mathbf{s})$ is arbitrary.

*4. **Example.*** Consider Eq.(1.1) with $V(u) = -\alpha^2(x)\cos u$, $\alpha > 0$, $m = 1$. The soluton is determined by formula (1.3), and the point $x_0$ satisfies the condition $\partial\alpha/\partial x\,|_{x=x_0} = 0$.

Eq.(1.1) with $m = 2$ occurs in the two-dimensional theory of elasticity of nematic liquid crystals (NLC), in which case $V = -{}^1/_2\alpha^2(x)\cos 2u$, the function $u(x, y)$ determines the mutual angle of orientation of the magnetic field **H** and the director **l**. The coefficient $\alpha^2(x)$ equals $\chi_a K^{-1}\mathbf{H}^2$, where **H** is the magnetic field strength, $\chi_a$ is the diamagnetic susceptibility of the crystal, and we are confining our attention to NLC having equal values $K$ of the moduli of longitudinal and transverse bending /9, 10/.

Let us assume that the field **H** is intensive: $\mathbf{H}^2 = \omega^2 h^2(x, y)$, $h_x'$, $h_y' = O(1)$, $\omega \gg 1$. Then we can construct asymptotic solutions as in Sect.2. These solutions have the sense of domain walls in NLC, and the hypersurfaces $S$ are simply curves satisfying the condition

$$\delta \int_S \alpha(x(s), y(s))\, ds = 0 \tag{4.1}$$

This is formally the same as Fermat's principle of geometrical optics for a medium in which the speed of light is $c(x, y) = \alpha^{-1}(x, y)$, though the physical meaning of the solution is quite different. This analogy shows that the curves $S$ may be determined using standard methods /11/. The principal term of the asymptotic expansion is given by a formula similar to (1.3).

The solution just constructed breaks down near points where the curve $S$ intersects itself.

*5. **The case of Eq.(1.2).*** It might seem that the existence of a simple variational principle for concentrated solutions is largely due to the fact that the principal term $u_0$ can be determined explicitly. That this is not the case can be shown by considering Eq.(1.2). We shall present only the main results, since the constructions are analogous to those in Sects.2 and 3; they may be found in /12, 13/.

We will seek a solution of Eq.(1.2), where $\mathbf{x} \in R^3$, concentrated near a curve $l$. In the neighbourhood of $l$ we introduce coordinates $n$, $p$, $s$, where $n$ is the distance of the point from the curve along the normal, $p$ is its distance along the binormal, and $s$ is a natural parameter on the curve. The principal term of the asymptotic expansion is

$$u = U_0(\nu, s)\exp\left(i\omega\int_{S_0}^{S} q(s')\,ds'\right),\quad q = q_0 + \omega^{-1}q_1,\quad \nu = \sqrt{n^2+p^2} \tag{5.1}$$

where $U_0, q_0, q_1$ are real. The function $U_0$ satisfies the equation

$$U_{0\nu\nu}'' + \nu^{-1}U_{0\nu}' + \Phi_{|u|^2}(U_0^2, 0, s) - q_0^2(s)U_0 = 0 \tag{5.2}$$
$$\Phi(U_0^2, s) = \Phi(U_0^2, n, p, s)|_{n=0,\, p=0}$$

For a fairly large class of potentials $\Phi$, there exists a non-trivial solution of Eq. (5.2) which decreases exponentially as $\nu \to \infty$ /13/. This is the case, e.g. for the potentials defined by (1.4).

As in Sect.2, orthogonality conditions of type (2.9) imply conditions on curve $l$ and enable us to determine the principal part of the phase $q_0(s)$. We have

$$q_0(s)I(s) = C = \text{const},\quad I(s) = \int_0^\infty \nu U_0^2(\nu, s)\,d\nu$$
$$(2q_0^2 I - K)\varkappa = \partial F/\partial n\,|_{n=p=0},\quad \partial F/\partial p\,|_{n=p=0} = 0 \tag{5.3}$$
$$K = \int_0^\infty \nu U_{0\nu}^2(\nu, s)\,d\nu,\quad F = \int_0^\infty \nu\Phi(U_0^2(\nu, s), n, p, s)\,d\nu$$

It turns out that the system of Eqs. (5.3) determining curve $l$ can be derived from a variational principle:

$$\delta\int_l a(\mathbf{x}(s))\,ds = 0,\quad a(x) = 2Cq_0(\mathbf{x}) - K(\mathbf{x})$$

where $I, K, q_0, U_0$ are the values of $I, K, q_0, U_0$ defined by (5.2), (5.3), with the parameter $s$ replaced by $\mathbf{x}$, i.e., they are the quantities obtained from Eqs.(5.2), (5.3), extended from $l$ to the whole space $R^3$.

The variational principle (3.2) is a Fermat-type principle for non-linear wave beams. Unlike the conventional Fermat principle, it contains an additional parameter $C$. The appearance of this parameter is due to the existence of a relationship between the amplitude and the phase $q$. As $C$ increases, the intensity of the beam, defined by $q$, also increases. For beams of different intensity one obtains different curves $l$.

At first sight one might think that, since $U_0$ is not explicitly evaluated, the principle (3.2) is of little practical use and curve $l$ (the analogue of a ray for a non-linear medium) cannot be found without using numerical methods. However, if the dependence of $\Phi$ on x can be made explicit by a simple transformation, the non-linear Fermat principle becomes as simple to use as in the linear case, just as in the case of Eq.(1.1). A transformation of this sort is possible, e.g., for potentials of the form (1.4).

An example will suffice. Consider (1.4) with $p = 2$. The standard Eq.(5.2) can be simplified by the substitution

$$U_0 = c_1 v(\rho),\quad \rho = c_2\nu,\quad c_1 = (q_0^2 - \beta)^{1/2}\alpha^{-1/2},\quad c_2 = (q_0^2 - \beta)^{1/2}$$

Finally we obtain $a(\mathbf{x}) = C^2\alpha(\mathbf{x}) + \beta(\mathbf{x})\alpha^{-1}(\mathbf{x})$. Thus, the problem of determining the trajectory of the beam may be considered, as in the linear case, apart from that of determining the entire structure of the beam - one need only find curve $l$, without knowing the detailed structure of the beam.

The author is indebted to I.A. Molotkov and E.L. Aero for their advice and comments.

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Translated by D.L.

*PMM U.S.S.R.*, Vol.53, No.4, pp.500-507, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00


# EDGE EFFECT IN THE BENDING OF A THIN THREE-DIMENSIONAL PLATE*

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The boundary layer near the rigidly clamped edge of a thin three-dimensional plate subjected to bending loads is investigated. It is shown that taking account of the next term in the deflection asymptotic form results in the appearance of inhomogeneities in the boundary conditions on the plate edge. It is proved that far from the edge the difference in the solution of the problem in an invariant formulation and the three-dimensional solution is inversely proportional to the plate thickness (the error for the Kirchhoff solution is inversely proportional to the square of the thickness; near the edge the accuracies of both solutions is identical). A correction term is found in a representation of the eigenfrequencies of the bending vibrations and a comparison is made with the Reissner theory.

*1. Formulation of the problem.* Let $\Omega$ be a domain on the plane $\mathbf{R}^2$ bounded by a closed simple smooth (class $C^\infty$) contour $\partial\Omega$, $Q$ is a cylinder $\{x\colon y = (x_1, x_2) \in \Omega,\ |x_3| < {}^1/_2 h\}$ of low altitude $h$ with side surface $S_h$ and bases $\Gamma_h^\pm$. We examine the three-dimensional problem of elasticity theory

$$\mu\Delta \mathbf{u}(h, \mathbf{x}) + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}(h, \mathbf{x}) + h^{-1} f(y)\,\mathbf{e}^{(3)} = 0, \quad \mathbf{x} \in Q_h \tag{1.1}$$

$$\boldsymbol{\sigma}^{(3)}(\mathbf{u}; h, \mathbf{x}) = p_\pm(\mathbf{y})\,\mathbf{e}^{(3)}, \quad \mathbf{x} \in \Gamma_h^\pm \tag{1.2}$$

$$\mathbf{u}(h, \mathbf{x}) = 0, \quad \mathbf{x} \in S_h \tag{1.3}$$

*Prikl. Matem. Mekhan., 53, 4, 642-650, 1989